Chapter 9

DETERMINANTS

Let us consider a pair of simultaneous equations:

(A) $2x - 3y = 9$
(B) $8x + 5y = 7$

We seek numbers x and y which satisfy (A) and (B) simultaneously, that is, a pair x,y for which both (A) and (B) become true statements. Such a pair also satisfies the equations

(A') $10x - 15y = 45$
(B') $24x + 15y = 21$

obtained by respectively multiplying both sides of (A) by 5 and both sides of (B) by 3. A pair x, y satisfying both (A') and (B') has to satisfy

(C) $34x = 66$

which is obtained by adding corresponding sides of (A') and (B'). The only root of (C) is $x = 66/34 = 33/17$.

If one replaces x by 33/17 in (A), one finds that

$$\begin{aligned} 2(33/17) - 3y &= 9 \\ 66/17 - 3y &= 9 \\ (66/17) - 9 &= 3y \\ -87/17 &= 3y \\ -29/17 &= y \end{aligned}$$

Hence the only pair of numbers x,y which might satisfy (A) and (B) simultaneously is $x = 33/17$, $y = -29/17$. Our method of obtaining these numbers shows that they do satisfy (A). We check that they also satisfy (B) by substituting in the left side as follows:

$$8\cdot\frac{33}{17} + 5\cdot\frac{-29}{17} = \frac{264}{17} + \frac{-145}{17} = \frac{119}{17} = 7$$

This shows that $x = 33/17$, $y = -29/17$ is the unique pair that satisfies (A) and (B) simultaneously.

An equation of the form $ax + by = c$ in which a, b and c are known numbers is called a **first-degree equation** in x and y, or a **linear equation** in x and y. Thus (A) and (B) are an example of a pair of **simultaneous linear equations**. The method illustrated above for solving such a pair is called **elimination**. More specifically, we eliminated y to obtain equation (C).

Eliminating y (or x) from the simultaneous equations

(D) $\qquad 6x - 15y = -10$
(E) $\qquad 4x - 10y = -7$

leads, upon multiplying both sides of (D) by 2 and both sides of (E) by -3, to

(D') $\qquad 12x - 30y = -20$
(E') $\qquad -12x + 30y = 21$

and, upon adding corresponding sides of (D') and (E'), to

(F) $\qquad 0 = 1$

Assuming that there is a pair x, y satisfying (D) and (E) simultaneously has led us to the false conclusion that $0 = 1$. Hence there is no pair x, y which satisfies (D) and (E) simultaneously.

9.1 DETERMINANTS OF ORDER 2

Let us now examine the general pair of simultaneous first-degree equations:

$$\text{(G)} \qquad \begin{aligned} a_1x + b_1y &= c_1 \\ a_2x + b_2y &= c_2 \end{aligned}$$

We are going to apply the elimination technique discussed above to (G) and then introduce the related concept of a determinant, which has important applications in the theory of systems of equations and in other fields.

If we multiply both sides of the first equation in (G) by b_2 and both sides of the second equation by b_1, we obtain

$$\begin{aligned} a_1b_2x + b_1b_2y &= c_1b_2 \\ a_2b_1x + b_1b_2y &= c_2b_1 \end{aligned}$$

Whence, by subtraction, we obtain:

(1) $\qquad (a_1b_2 - a_2b_1)x = c_1b_2 - c_2b_1.$

If we use the equations in (G) to eliminate x rather than y, the result is

(2) $\qquad (a_1b_2 - a_2b_1)y = a_1c_2 - a_2c_1.$

If $a_1b_2 - a_2b_1$ is not zero, we see that the solution of the system (G) is found from (1) and (2) in the form

(3) $$x = \frac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1}, \qquad y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$$

We note that the denominators are the same, that the numerator in the expression for x is obtained from the denominator by replacing the coefficients a_1 and a_2 of x in (G) with c_1 and c_2, respectively, and that the numerator in the expression for y is obtained from the denominator by replacing the coefficients b_1 and b_2 of y with c_1 and c_2.

This motivates us to introduce the notation

(4) $$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

The square array bordered by vertical lines in (4) is called a two-by-two (2x2) **determinant**. With this notation, the equations in (3) can be rewritten as

(5) $$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, \qquad y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

provided that the common denominator is not zero.

Thus we see that the solution of a system of simultaneous first-degree equations can be written as ratios of determinants in which the denominator is the determinant made up of coefficients of x and y in the order in which they appear in the equations, while the numerator for x is the same determinant with the coefficients of x replaced by the constants, and the numerator for y is the determinant of the denominator with the coefficients of y replaced by the constants. This technique is called **Cramer's Rule**. The common denominator is called the **determinant of the system**.

Example 1. Solve by determinants:

$$3x + 2y = 5$$
$$x - 7y = 2$$

Solution: We first evaluate the determinant in the denominator (the determinant of the system), as follows:

$$\begin{vmatrix} 3 & 2 \\ 1 & -7 \end{vmatrix} = 3(-7) - 1\cdot 2 = -21 - 2 = -23$$

Since this determinant is not zero, the system has the unique solution:

$$x = \frac{\begin{vmatrix} 5 & 2 \\ 2 & -7 \end{vmatrix}}{-23} = \frac{5(-7) - 2\cdot 2}{-23} = \frac{-35 - 4}{-23} = \frac{-39}{-23} = \frac{39}{23}$$

$$y = \frac{\begin{vmatrix} 3 & 5 \\ 1 & 2 \end{vmatrix}}{-23} = \frac{3\cdot 2 - 1\cdot 5}{-23} = \frac{6 - 5}{-23} = \frac{1}{-23} = -\frac{1}{23}.$$

Example 2. Use determinants to investigate the simultaneous equations:

$$\text{(H)} \qquad \begin{aligned} 10x - 14y &= 5 \\ 15x - 21y &= 8 \end{aligned}$$

Solution: The determinant of the system is

$$D = \begin{vmatrix} 10 & -14 \\ 15 & -21 \end{vmatrix} = 10(-21) - 15(-14) = -210 + 210 = 0.$$

Since $D = 0$, we cannot solve for x and y in the form of (3) above. However, we can consider forms (1) and (2), since they do not involve division by zero. Evaluating the other determinants, we have

$$\begin{vmatrix} 5 & -14 \\ 8 & -21 \end{vmatrix} = 5(-21) - 8(-14) = -105 + 112 = 7$$

$$\begin{vmatrix} 10 & 5 \\ 15 & 8 \end{vmatrix} = 10\cdot 8 - 15\cdot 5 = 80 - 75 = 5.$$

Thus (1) and (2) lead to

$$\text{(I)} \qquad \begin{aligned} 0 &= 0\cdot x = 7 \\ 0 &= 0\cdot y = 5. \end{aligned}$$

Because the contradictory equations (I) are implied by the system (H), this latter system has no simultaneous solution.

It can be shown that if the determinant of system (G) is zero, then (G) has no solution or an infinite number of solutions.

9.2 DETERMINANTS OF ORDER 3

When the elimination technique is applied to the general system

(1)
$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}$$

of first-degree equations in the three unknowns *x, y,* and *z,* one obtains equations of the form

(2)
$$Dx = E, \quad Dy = F, \quad Dz = G$$

where *D* is the three-by-three determinant (or determinant of order 3) defined by

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1b_2c_3 - a_1b_3c_2 + a_2b_3c_1 - a_2b_1c_3 + a_3b_1c_2 - a_3b_2c_1.$$

In the equations above, *E, F,* and *G* are obtained by substituting the column of *d*'s for the column of *a*'s, *b*'s, or *c*'s, respectively, in *D*. For example,

$$F = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1d_2c_3 - a_1d_3c_2 + a_2d_3c_1 - a_2d_1c_3 + a_3d_1c_2 - a_3d_2c_1.$$

If $D \neq 0$, it follows from the equations (2) above that the system of simultaneous equations (1) has the unique solution

(3)
$$x = \frac{E}{D}, \; y = \frac{F}{D}, \; z = \frac{G}{D}.$$

As in the case of 2 by 2 determinants, it can be shown that if $D = 0$, then the simultaneous equations (1) either have no common solution or an infinite number of common solutions. The determinant *D* is called the **determinant of the system**. The technique of expressing the solution of a system of simultaneous linear equations in terms of ratios of determinants when the determinant of the system is not zero, as in (3) above, is called **Cramer's Rule**.

We note that, according to the definition

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1b_2c_3 - a_1b_3c_2 + a_2b_3c_1 - a_2b_1c_3 + a_3b_1c_2 - a_3b_2c_1$$

given above, the three-by-three determinant D consists of a sum of products of the form $\pm\ a_ib_jc_k$ where i, j, k is a permutation of 1, 2, 3, and the plus sign is chosen when the permutation is even and the minus sign when it is odd. (For a definition of even and odd permutations, see Chapter 7.) Since there is a term corresponding to each permutation, the number of terms is 3! = 6, half preceded by a plus sign and half by a minus sign. (See Problem 21, Chapter 7.) Is should be noted that these observations also apply to two-by-two determinants

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$$

in that here the permutation 1, 2 is even and the permutation 2, 1 is odd, so that the 2! = 2 terms are preceded by the appropriate signs.

Example. Evaluate the determinant of the following system and thus show that the system has a unique solution:

$$\begin{aligned} 2x - 3z &= 10 \\ 5x + 4y &= 11 \\ y - 6z &= -3 \end{aligned}$$

Solution:

$$D = \begin{vmatrix} 2 & 0 & -3 \\ 5 & 4 & 0 \\ 0 & 1 & -6 \end{vmatrix}$$

$$\begin{aligned} &= 2\cdot4\cdot(-6) - 2\cdot1\cdot0 + 5\cdot1\cdot(-3) - 5\cdot0\cdot(-6) + 0\cdot0\cdot0 - 0\cdot4\cdot(-3) \\ &= -48 - 15 = -63 \end{aligned}$$

Since $D \neq 0$, there is a unique solution.

We now introduce a double subscript notation for the entries in a determinant, with the first subscript giving the row and the second giving the column. In this notation, a determinant of order 3 can be written as

(4) $$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

and the expanded value is

(5) $$D = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}.$$

Each term in (5) is of the form $\pm a_{1i}a_{2j}a_{3k}$ with the plus sign used if *i*, *j*, *k* is an even permutation of 1, 2, 3 and the minus sign if *i*, *j*, *k* is an odd permutation. The terms in (5) can be grouped in several ways, one of which is

(6) $$D = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}).$$

The coefficient $a_{21}a_{32} - a_{22}a_{31}$ of a_{13} in (6) is the value of the 2 by 2 determinant that results when the entire first row and third column of D are removed; the coefficients of a_{11} and $-a_{12}$ in (6) can be characterized similarly.

This motivates the following definitions: Let a_{ij} be a given entry in the determinant D of (4) and let M_{ij} be the 2 by 2 determinant obtained by deleting the *i*th row and the *j*th column of D. This determinant M_{ij} is called the **minor** of the entry a_{ij}. The **cofactor** C_{ij} of the entry a_{ij} is defined by the formula

$$C_{ij} = (-1)^{i+j}M_{ij}.$$

For example, the minor of a_{23} is

$$M_{23} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} = a_{11}a_{32} - a_{12}a_{31}$$

and the cofactor of a_{23} is

$$C_{23} = (-1)^{2+3}M_{23} = -(a_{11}a_{32} - a_{12}a_{31}).$$

It now can easily be seen that equation (6) may be rewritten as

$$D = a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13}$$

or as

$$D = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}.$$

Problems for Sections 9.1 and 9.2

1. Solve the following system by determinants, that is, by using Cramer's rule.

$$9x - 7y = 11$$
$$5x + 2y = -4$$

2. Show by determinants that the following system has no solution.

$$15x + 20y = -13$$
$$21x + 28y = 5$$

3. Solve the system given in the example of Section 9.2.

4. Use Cramer's rule to solve the following system:

$$7x - 3y - z = 0$$
$$2x - 2y + z = 5$$
$$-x + y + 2z = 25$$

5. Use Cramer's rule to solve the following system:

$$x - y + 3z = 15$$
$$2x - 4y + z = 6$$
$$3x + 3y - 6z = -3$$

6. Solve the following system for x, y, and z:

$$2x + y = 0$$
$$y + 2z = 0$$
$$x + z = 0$$

7. Solve the following system for x, y, and z in terms of a, b, and c:

$$x + y = c$$
$$y + z = a$$
$$x + z = b$$

8. Show the following for the determinant

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

(a) $D = a_1M_1 - a_2M_2 + a_3M_3$, where M_1, M_2, and M_3 are the minors of a_1, a_2, and a_3 respectively.

(b) $D = a_2A_2 + b_2B_2 + c_2C_2$, where A_2, B_2, and C_2 are the cofactors of a_2, b_2, and c_2, respectively.

9. Show the following for the determinant

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}.$$

(a) $a_{31}M_{21} - a_{32}M_{22} + a_{33}M_{23} = 0$.

(b) $a_{11}C_{13} + a_{21}C_{23} + a_{31}C_{33} = 0$.

10. Show the following:

(a) $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$.

(b) $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$.

11. (a) Show that $\begin{vmatrix} a_1 & b_1 & c_1 \\ ka_2 & kb_2 & kc_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = k\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$.

(b) Show that if each element of a fixed row of a three by three determinant D is multiplied by a factor k, the new determinant equals kD.

12. (a) Evaluate $\begin{vmatrix} 0 & 0 & 0 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$.

(b) Show that if a three by three determinant D has a row of zeros, then $D = 0$.

(c) Show that if a three by three determinant D has a column of zeros, then $D = 0$.

13. Show the following:

(a) $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = -\begin{vmatrix} b_1 & a_1 \\ b_2 & a_2 \end{vmatrix}.$

(b) $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = -\begin{vmatrix} a_2 & b_2 \\ a_1 & b_1 \end{vmatrix}.$

(c) $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = -\begin{vmatrix} a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \\ a_2 & b_2 & c_2 \end{vmatrix}.$

(d) $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = -\begin{vmatrix} c_1 & b_1 & a_1 \\ c_2 & b_2 & a_2 \\ c_3 & b_3 & a_3 \end{vmatrix}.$

(e) $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = -\begin{vmatrix} b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$

14. Evaluate $\begin{vmatrix} a & b & c \\ a & b & c \\ d & e & f \end{vmatrix}.$

15. Show that in a three by three determinant if the elements of one row are a constant k times the elements of another row, then the determinant equals zero.

16. (a) Use the definition of a determinant to show that

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 + a_2' & b_2 + b_2' & c_2 + c_2' \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2' & b_2' & c_2' \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

(b) Show that if the elements of a given row of a three by three determinant D are $f_1 + g_1$, $f_2 + g_2$, $f_3 + g_3$, then $D = D_1 + D_2$ where D_1 results from D by replacing the given row by f_1, f_2, f_3 and D_2 by replacing the given row by g_1, g_2, g_3.

17. Show that if two rows of a three by three determinant D are u_1, u_2, u_3 and v_1, v_2, v_3, respectively, and if D^* is the same as the determinant D except that the row u_1, u_2, u_3 is replaced by $u_1 + kv_1$, $u_2 + kv_2$, $u_3 + kv_3$ where k is a constant, then $D^* = D$.

9.3 DETERMINANTS OF ORDER *n*

We have defined the 2 by 2 determinant

(1)
$$D = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

to be the number $ad - bc$ obtained from the square array

(2)
$$\begin{matrix} a & b \\ c & d \end{matrix}$$

of 4 numbers in two rows and two columns. Thus, bordering the array (2) with vertical lines converts the array into a symbol for the number D. Similarly, a 3 by 3 determinant is a number obtained in a certain manner from a square array of 9 numbers.

Our next objective is to define an n by n determinant. More precisely, we seek an unambiguous rule for obtaining a number from an n by n square array of numbers and want this rule to agree with the previous definitions when $n = 2$ or 3. Let

(3)
$$\begin{matrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \cdots & & & & \\ \cdots & & & & \\ \cdots & & & & \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{matrix}$$

be an array of n^2 numbers a_{ij}, where the first subscript designates the row and the second designates the column.

There are n! products

$$a_{1i}a_{2j}a_{3h}\cdots a_{nk}$$

with exactly one factor from each row and exactly one factor from each column. The determinant D associated with the array (3) is the sum of the n! terms

$$\pm a_{1i}a_{2j}a_{3h}\cdots a_{nk}$$

where the plus sign is used when the permutation

$$i, j, h, \ldots , k$$

is even, and the minus sign is used when the permutation is odd. As before, the array (3) is bordered with vertical lines in writing the symbol

(4)
$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \cdots & & & & \\ \cdots & & & & \\ \cdots & & & & \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix}$$

for a determinant D of order n.

The ***minor*** M_{ij} of the entry a_{ij} in D is the determinant of order n - 1 obtained by deleting the ith row and jth column of D. The ***cofactor*** of a_{ij} is defined by

$$C_{ij} = (-1)^{i+j}M_{ij}\,.$$

Associated with the determinant D of (4) is the determinant

$$D' = \begin{vmatrix} a_{11} & a_{21} & a_{31} & \cdots & a_{n1} \\ a_{12} & a_{22} & a_{32} & \cdots & a_{n2} \\ a_{13} & a_{23} & a_{33} & \cdots & a_{n3} \\ \cdots & & & & \\ a_{1n} & a_{2n} & a_{3n} & \cdots & a_{nn} \end{vmatrix}$$

in which the entries in the ith column are the corresponding entries in the ith row of D. The determinant D' is called the **transpose** of D; it is easily seen that D is then the transpose of D'. For example

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \text{ and } \begin{vmatrix} a & d & g \\ b & e & h \\ c & f & i \end{vmatrix}$$

are transposes of each other.

The following theorem enables one to prove results involving columns of a determinant from the corresponding results on the rows and vice versa.

THE TRANSPOSE THEOREM: Let D' be the transpose of a determinant D. Then $D' = D$.

Proof: Let a_{rs} and b_{rs} be the entries of D and D', respectively, in the rth row and sth column. Since D' is the transpose of D, $b_{rs} = a_{sr}$. By definition, D is the sum of $n!$ terms

(5) $$\pm\, a_{1i}a_{2j}\cdots a_{nh}$$

where the plus sign is used if the permutation

$$i, j, \ldots, h$$

is even and the minus sign if it is odd. Also, D' is the sum of $n!$ terms

(6) $$\pm\, b_{1x}b_{2y}\cdots b_{nz}$$

where the sign is plus if

(7) $$x,\ y,\ \ldots,\ z$$

is an even permutation, and minus otherwise. Since $b_{rs} = a_{sr}$, each term (6) can be rewritten as

(8) $$\pm\, a_{x1}a_{y2}\cdots a_{zn}$$

The terms (8) are all the products, with signs attached, in which there is exactly one factor from each row and from each column of D; hence the terms (8) are the terms of the expansion of D, except that the signs may not agree. We will therefore prove that $D = D'$ by showing that (8) has the same sign as a term of the expansion of D that it has in the expansion of D'. We do this by describing a method of determining the sign whether or not the row numbers of the entries are in the order $1, 2, \ldots, n$. Let

(9) $$\pm\, a_{pu}a_{qv}\cdots a_{rw}$$

be a term of the expansion of D with its factors in any order. Then the row subscript numbers and the column subscript numbers give us the two permutations:

(10) $$\begin{matrix} p, & q, & \ldots, & r \\ u, & v, & \ldots, & w \end{matrix}$$

We wish to prove that the plus sign should be used in (9) if both of the permutations of (10) are even or both are odd, and that the minus sign should be used when one is even and the other odd. When the entries in (9) have their row numbers in the normal order, the permutations (10) are of the form

$$\begin{matrix} 1, 2, \ldots, n \\ i, j, \ldots, h \end{matrix}$$

with the top one even, and hence the new 2-permutation rule indicates that the sign should be plus when the bottom permutation is even and minus otherwise. This agrees with the definition of a determinant and shows that the new rule is correct in this case.

We can go from the order $a_{1i}a_{2j}\ldots a_{nh}$ of the factors to any order $a_{pu}a_{qv}\ldots a_{rw}$ by means of a number of interchanges of adjacent factors. Whenever one such interchange is made, the row subscript permutation and column subscript permutation will each change from even to odd or from odd to even. This means that the new rule will continue to indicate the correct sign as these interchanges are made. We are especially interested in the case in which the column numbers are in order, that is, the case in which (9) is (8)

$$\pm\, a_{x1}a_{y2}\cdots a_{zn}.$$

The permutations (10) then become

(11) $$\begin{matrix} x, & y, & \ldots, & z \\ 1, & 2, & \ldots, & n \end{matrix}$$

with the bottom one even. Hence the 2-permutation rule indicates that the plus sign is used if and only if the permutation $x, y, \ldots, z$ of (11) is even. This is exactly the rule for determining the sign of (8)

$$\pm\, a_{x1}a_{y2}\cdots a_{zn}$$

as a term of D'. Hence the sign for (8) is the same either as a term of D' or of D, since the sign in both cases agrees with the 2-permutation rule. This shows that $D' = D$ and completes the proof.

Let

(12) $$\begin{matrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n = b_3 \\ \cdots \\ a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{nn}x_n = b_n \end{matrix}$$

be a system of n simultaneous first-degree equations in n unknowns, $x_1, x_2, \ldots, x_n$. The determinant D of (4) is the **determinant of the system** for (12).

Let $D_1, D_2, \ldots, D_n$ be the determinants that result when the column of b's in (12) is substituted for the first, second, ..., nth column, respectively, of the D of (4). The general **Cramer's Rule** states that if $D \neq 0$, then the system (12) has the unique solution

(13) $$x_1 = \frac{D_1}{D},\ x_2 = \frac{D_2}{D},\ \ldots,\ x_n = \frac{D_n}{D}$$

and that if $D = 0$, then the system either has no solution or an infinite number of solutions. We do not give the proof of this rule for general n.

When the number of equations in (12) is large, it becomes very difficult to evaluate the $n + 1$ determinants in (13) by the methods discussed in this book. More advanced texts describe variations of the elimination techniques that are practical for the numerical approximation of determinants of large order or in the solution of systems (12) with n large. (For example, see the description of Crout's Method in the appendix of F. B. Hildebrand's *Methods of Applied Mathematics*, Prentice-Hall, 1952.)

Problems for Section 9.3

In Problems 1-7, below, D represents a determinant of order n. Prove each statement either from the definition of an n by n determinant, by using the Transpose Theorem, or by using previous results.

R 1. If all the entries on a given row (or column) of D are multiplied by a fixed number k, the value of D is multiplied by k.

R 2. If each entry in a given row (or column) of D is zero, then $D = 0$.

R 3. (a) If any two columns of a determinant D are interchanged, the resulting determinant D_1 equals $-D$. (See Problem 21, Chapter 7.)

(b) If any two rows of a determinant D are interchanged, the resulting determinant D_2 equals $-D$.

R 4. If the entries of a row (or column) of D are a constant k times the corresponding entries of another row (or column), then $D = 0$.

R 5. If the entries of a given row (or column) of D are $f_1 + g_1$, $f_2 + g_2$, ..., $f_n + g_n$, then $D = D_1 + D_2$, where D_1 results from D by replacing the given row (or column) by $f_1, f_2, \ldots, f_n$ and D_2 by replacing the given row (or column) by $g_1, g_2, \ldots, g_n$.

R 6. Let $u_1, u_2, \ldots, u_n$ and $v_1, v_2, \ldots, v_n$ be the entries of two rows (or columns) of D, and let D^* result from replacing $v_1, v_2, \ldots, v_n$ in D by $v_1 + ku_1, v_2 + ku_2, \ldots, v_n + ku_n$, respectively. Then $D^* = D$.

R 7. Let a_{ij} be the entry in the ith row and jth column of D.

(a) If S is the sum of all the terms of the expansion of D that involve a_{nn}, then $S = a_{nn}M_{nn} = a_{nn}C_{nn}$, where M_{nn} and C_{nn} are the minor and cofactor of a_{nn}. (See Problem 22, Chapter 7.)

(b) Let T be the sum of all the terms of the expansion of D that involve a fixed entry a_{hk}. Then T = $(-1)^{h+k}a_{hk}M_{hk} = a_{hk}C_{hk}$. (Use Problem 3, above, and Part (a) of this problem.)

(c) If h is one of the numbers 1, 2, ..., n, then each term of the expansion of D has one and only one of the entries a_{h1}, a_{h2}, ..., a_{hn} as a factor.

(d) $D = (-1)^{h+1}a_{h1}M_{h1} + (-1)^{h+2}a_{h2}M_{h2} + \cdots + (-1)^{h+n}a_{hn}M_{hn}$.

(e) $D = a_{h1}C_{h1} + a_{h2}C_{h2} + \cdots + a_{hn}C_{hn}$.

(f) If k is any one of the numbers 1, 2, ..., n then $D = a_{1k}C_{1k} + a_{2k}C_{2k} + \cdots + a_{nk}C_{nk}$.

8. (a) Find specific numbers a, b, c, and d such that the polynomial $f(x) = ax^3 + bx^2 + cx + d$ has the values listed in the table below. Check, using $f(5) = 165$.

x	1	2	3	4
$f(x)$	1	10	35	84

(b) Follow the instructions of the previous part for the table below. Check, using $f(4) = 30$.

x	0	1	2	3
$f(x)$	0	1	5	14

9. (a) Show that

$$\begin{vmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{vmatrix} = a_{11}a_{22}$$

and that

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33}.$$

(b) Let D be an n by n determinant, with the element a_{ij} in the ith row and jth column 0 if $i > j$. Show that $D = a_{11}a_{22}a_{33}...a_{nn}$.

10. (a) Evaluate $\begin{vmatrix} 1 & a \\ a & 1 \end{vmatrix}$ and $\begin{vmatrix} 1 & a & a^2 \\ a & 1 & a \\ a^2 & a & 1 \end{vmatrix}$.

(b) Show that $\begin{vmatrix} 1 & a & a^2 & a^3 \\ a & 1 & a & a^2 \\ a^2 & a & 1 & a \\ a^3 & a^2 & a & 1 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 & a^3 \\ 0 & 1-a^2 & a-a^3 & a^2-a^4 \\ 0 & 0 & 1-a^2 & a-a^3 \\ 0 & 0 & 0 & 1-a^2 \end{vmatrix}$.

(c) Evaluate $\begin{vmatrix} 1 & a & a^2 & a^3 & a^4 \\ a & 1 & a & a^2 & a^3 \\ a^2 & a & 1 & a & a^2 \\ a^3 & a^2 & a & 1 & a \\ a^4 & a^3 & a^2 & a & 1 \end{vmatrix}$.

(d) In the determinants of Parts (a), (b), and (c) of this problem, the element c_{ij} in the ith row and jth column is $a^{|i-j|}$. Establish a compact formula for the value of the n by n determinant with $c_{ij} = a^{|i-j|}$. (See Section 8.4 for a definition of |x|.)

11. Show that $\begin{vmatrix} r-s & s-t & t-r \\ s-t & t-r & r-s \\ t-r & r-s & s-t \end{vmatrix} = 0$.

12. Show that $\begin{vmatrix} x_2+x_3 & x_1+x_3 & x_1+x_2 \\ y_2+y_3 & y_1+y_3 & y_1+y_2 \\ z_2+z_3 & z_1+z_3 & z_1+z_2 \end{vmatrix} = 2\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix}$.

13. Show that $\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix} = a^3 + b^3 + c^3 - 3abc$.

14. Show that $\begin{vmatrix} a & b & c & d \\ d & a & b & c \\ c & d & a & b \\ b & c & d & a \end{vmatrix} = (a+b+c+d)(a-b+c-d)[(a-c)^2 + (b-d)^2]$.

15. Evaluate $\begin{vmatrix} x+y & z & z \\ x & y+z & x \\ y & y & z+x \end{vmatrix}$.

16. Evaluate

(a) $\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{vmatrix}$.

(b) $\begin{vmatrix} 1+a & 1+a & 1+a & 1+a \\ 1+a & a & a & a \\ 1+a & a & 1+a & a \\ 1+a & a & a & 1+a \end{vmatrix}$.

17. Let F_0, F_1, F_2, ... be the Fibonacci sequence.

(a) Show that $\begin{vmatrix} F_{n+2} & F_{n+1} \\ F_{n+1} & F_n \end{vmatrix} = -\begin{vmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{vmatrix}$.

(b) Find numbers x, y, and z such that $F_{2n} = xF_n^2 + yF_nF_{n+1} + zF_{n+1}^2$.

(c) Find x, y, z, and w such that $F_{3n} = xF_n^3 + yF_n^2F_{n+1} + zF_nF_{n+1}^2 + wF_{n+1}^3$.

(d) Find analogues of the formulas above for the Lucas numbers.

18. Evaluate:

(a) $\begin{vmatrix} 1 & x & y & z+w \\ 1 & y & z & w+x \\ 1 & z & w & x+y \\ 1 & w & x & y+z \end{vmatrix}$.

(b) $\begin{vmatrix} x & y & z & w \\ y & x & z & w \\ y & x & w & z \\ x & y & w & z \end{vmatrix}$.

19. Let D be an n by n determinant with c_{ij} the entry in the ith row and jth column. Show that $D = 0$ if n is odd and $c_{ij} + c_{ji} = 0$ for all i and j.

20. Evaluate the n by n determinant with the entry c_{ij} in the ith row and jth column satisfying each of the following conditions: (It may be helpful to begin with small values of n and to try to find a pattern which suggests a proof.)

(a) $c_{ij} = \binom{i+j-2}{j-1}$.
(b) $c_{ij} = c_{1j}$ if $i > j$.
(c) $c_{ij} = a + |i - j|d$. (See Section 8.4 for a definition of |x|.)
(d) $c_{ij} = 1$ if $j - i$ is -1, 0, or a positive even integer, and $c_{ij} = 0$ for other values of $j - i$.
(e) $c_{ij} = a + x$ if $j > i$, $c_{ij} = b + x$ if $j < i$, and $c_{ii} = r_i + x$.
(f) $c_{ij} = \dfrac{1}{(j+2-i)!}$ for $i < j + 2$ and $c_{ij} = 0$ for $i \geq j + 2$.

9.4 VANDERMONDE AND RELATED DETERMINANTS

Determinants in which the elements of each column (or row) are the terms 1, r, r^2, ..., r^{n-1} of a geometric progression are called **<u>Vandermonde</u> <u>determinants</u>**, named for Alexandre-Théophile Vandermonde (1735-1796), who was the first to give a systematic treatment of the theory of determinants.

Let us evaluate the 4 by 4 Vandermonde determinant

$$D = \begin{vmatrix} 1 & 1 & 1 & 1 \\ x & a & b & c \\ x^2 & a^2 & b^2 & c^2 \\ x^3 & a^3 & b^3 & c^3 \end{vmatrix}.$$

The expansion by minors of first column entries as outlined in Problem 7, Section 9.3, yields the following:

$$D = \begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} - x\begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} + x^2\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} - x^3\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}.$$

Letting r, s, t, and u stand for the 3 by 3 determinants in this expression, we may write

$$D = f(x) = r - sx + tx^2 - ux^3.$$

If we let $x = a$ in D, two columns become identical and, by Problem 4, Section 9.3, D becomes zero. This means that $f(a) = 0$, and it follows from the Factor Theorem that $x - a$ is a factor of $f(x)$. Similarly, $x - b$ and $x - c$ are factors of $f(x)$.

If two of the numbers a, b, and c are equal, D has identical columns and thus is zero. We therefore assume that a, b, and c are distinct numbers. Then $f(x)$ is a multiple of the product of $x - a$, $x - b$, and $x - c$. Since $f(x)$ is a polynomial of degree 3 or less and has $-u$ as the coefficient of x^3, this means that $f(x)$ must be $-u(x - a)(x - b)(x - c)$. This can be written as

$$u(a - x)(b - x)(c - x).$$

We leave it as an exercise for the reader (in Problem 1 below) to show that the 3 by 3 determinant u is expressible as $(b - a)(c - a)(c - b)$. Substituting this for u in the above gives the result:

$$D = (a - x)(b - x)(c - x)(b - a)(c - a)(c - b).$$

Problems for Section 9.4

R 1. Show that $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (b - a)(c - a)(c - b).$

2. Show that $\begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix} = (b - a)(c - a)(c - b)(a + b + c).$

3. Show that $\begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} = \begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ bc & ca & ab \end{vmatrix} = (b - a)(c - a)(c - b)(bc + ca + ab).$

4. Express $\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \end{vmatrix}$ as a product of 6 first-degree factors.

5. Express $\begin{vmatrix} 1 & -x & x^2 & -x^3 \\ 1 & -y & y^2 & -y^3 \\ 1 & -z & z^2 & -z^3 \\ 1 & -w & w^2 & -w^3 \end{vmatrix}$ as a product of 6 first-degree factors.

6. Evaluate $\begin{vmatrix} 1 & a+x \\ 1 & a+y \end{vmatrix}$ and $\begin{vmatrix} 1 & a+x & b+cx+x^2 \\ 1 & a+y & b+cy+y^2 \\ 1 & a+z & b+cz+z^2 \end{vmatrix}$ in factored form.

7. Evaluate $\begin{vmatrix} 1 & 2-x & 3+4x+x^2 & 5-x^3 \\ 1 & 2-y & 3+4y+y^2 & 5-y^3 \\ 1 & 2-z & 3+4z+z^2 & 5-z^3 \\ 1 & 2-w & 3+4w+w^2 & 5-w^3 \end{vmatrix}$ in factored form.

8. Show the following:

(a) $\begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & 2x \\ 1 & y & y^2 \end{vmatrix} = (y-x)^2.$

(b) $\begin{vmatrix} 1 & 0 & 0 & 1 \\ x & 1 & 0 & y \\ x^2 & 2x & 2 & y^2 \\ x^3 & 3x^2 & 6x & y^3 \end{vmatrix} = 2(y-x)^3.$

9. Find integers r, s, and t such that

$$\begin{vmatrix} 1 & 0 & 1 & 1 \\ x & 1 & y & z \\ x^2 & 2x & y^2 & z^2 \\ x^3 & 3x^2 & y^3 & z^3 \end{vmatrix} = (y-x)^r(z-x)^s(z-y)^t.$$

10. Show that $\begin{vmatrix} 1 & x & x^2 & x^3 & x^4 \\ 0 & 1 & 2x & 3x^2 & 4x^3 \\ 0 & 0 & 2 & 6x & 12x^2 \\ 1 & y & y^2 & y^3 & y^4 \\ 0 & 1 & 2y & 3y^2 & 4y^3 \end{vmatrix} = 2(y-x)^6.$

11. Find integers r, s, and t such that $\begin{vmatrix} 1 & 0 & 1 & 0 & 1 \\ x & 1 & y & 1 & z \\ x^2 & 2x & y^2 & 2y & z^2 \\ x^3 & 3x^2 & y^3 & 3y^2 & z^3 \\ x^4 & 4x^3 & y^4 & 4y^3 & z^4 \end{vmatrix} = (y-x)^r(z-x)^s(z-y)^t.$

12. Solve the following system of equations if $a < b < c$:

$$\begin{aligned} x + y + z &= 3 \\ ax + by + cz &= a + b + c \\ a^2x + b^2y + c^2z &= a^2 + b^2 + c^2. \end{aligned}$$

13. (a) Show that the system

$$\begin{aligned} x + y + z &= 3 \\ a^2x + b^2y + c^2z &= a^2 + b^2 + c^2 \\ a^3x + b^3y + c^3z &= a^3 + b^3 + c^3 \end{aligned}$$

has a solution if the a, b, c are distinct and $ab + bc + ca \neq 0$.

(b) Calculate the solution.

14. Evaluate:

(a) $\begin{vmatrix} 1 & a & a^2 + 2bc \\ 1 & b & b^2 + 2ca \\ 1 & c & c^2 + 2ab \end{vmatrix}.$

(b) $\begin{vmatrix} 1 & a & a^2 & a^3 + bcd \\ 1 & b & b^2 & b^3 + cda \\ 1 & c & c^2 & c^3 + dab \\ 1 & d & d^2 & d^3 + abc \end{vmatrix}.$

(c) $\begin{vmatrix} 1 & a & a^2 & a^3+2bcd \\ 1 & b & b^2 & b^3+2cda \\ 1 & c & c^2 & c^3+2dab \\ 1 & d & d^2 & d^3+2abc \end{vmatrix}.$

15. Let D be the general n by n Vandermonde determinant with the entry c_{ij} in the ith row and jth column given by $c_{ij} = a_j^{i-1}$. Show that D is the product

$$(a_2 - a_1)(a_3 - a_1)(a_3 - a_2)...(a_n - a_{n-1})$$

of all the different $a_r - a_s$ with $1 \leq s < r \leq n$.